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On Polynomials with Curved Majorants

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1. INTRODUCTION

Let $T_m(x) := \cos(m \arccos x)$ and $U_m(x) := (1-x^2)^{-1/2} \sin\{(m+1) \arccos x\}$ denote, as usual, the Chebyshev polynomials of the first and second kind, respectively, of degree m. Generalizing a classical result of W. A. Markov, it was proved in [5] that if λ, μ are non-negative integers and $P(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree at most n such that

$$|P(x)| \leqslant (1-x)^{\lambda/2} (1+x)^{\mu/2} \qquad \text{for} \quad -1 < x < 1,$$

then, for $(\lambda+\mu)/2 \leqslant j \leqslant n$,

$$\max_{-1 \leqslant x \leqslant 1} |P^{(j)}(x)| \leqslant \max\{ \max_{-1 \leqslant x \leqslant 1} |\Lambda_n^{(j)}(x)|, \max_{-1 \leqslant x \leqslant 1} |\Lambda_{n-1}^{(j)}(x)| \},$$

where

$$\Lambda_m(x) := \begin{cases} (1-x)^{\lambda/2} (1+x)^{\mu/2} T_{m-(\lambda+\mu)/2}(x) \\ \qquad \text{if } \lambda, \mu \text{ are both even} \\ (1-x)^{(\lambda+1)/2} (1+x)^{(\mu+1)/2} U_{m-1-(\lambda+\mu)/2}(x) \\ \qquad \text{if } \lambda, \mu \text{ are both odd.} \end{cases}$$

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The case $1 \leqslant j < (\lambda+\mu)/2$, for $(\lambda+\mu)/2 > 1$, was left unresolved. For example, the above result does not say anything about $\max_{-1 \leqslant x \leqslant 1} |P'(x)|$, if $\lambda = \mu = 2$. The present paper is mainly devoted to this particular problem. We shall also discuss the following related question which was raised by the late Professor P. Turán during a visit to the Université de Montréal in 1975.

QUESTION. *Given a polynomial P of degree at most n satisfying*

$$0 \leqslant P(x) \leqslant (1-x^2)^{1/2} \qquad \text{for} \quad -1 < x < 1,$$

how large can $\max_{-1 \leqslant x \leqslant 1} |P'(x)|$ *be?*

2. The Derivative of a Polynomial Whose Modulus is $\leqslant 1 - x^2$ on $(-1, 1)$

2.1. We find it advisable to introduce a few notations.

Let $\mathscr{P}_m$ be the set of all polynomials of degree at most m. We denote by F_m and F_m^* the subsets consisting of those $P \in \mathscr{P}_m$ for which

$$\|P\| := \max_{-1 \leqslant x \leqslant 1} |P(x)| \leqslant 1$$

and

$$\|P\|_* := \sup_{-1 < x < 1} \frac{|P(x)|}{1-x^2} \leqslant 1,$$

respectively.

2.2. First we prove the following proposition which will serve as a lemma.

PROPOSITION 1. *If $P \in F_n^*$ and $P(x)$ is real for real values of x, then*

$$\{P'(x)\}^2 + (n^2-4n)\left\{\frac{P(x)}{1-x^2}\right\}^2 \leqslant (n-2)^2 \qquad \text{for} \quad -1 \leqslant x \leqslant 1. \tag{1}$$

Proof. Clearly $P(x) = (1-x^2)\,q(x)$ where $q \in F_{n-2}$. Thus $P(\cos\theta) = (\sin^2\theta)\,t(\theta)$ where $t(\theta) = q(\cos\theta)$ is a real trigonometric polynomial of degree at most $n-2$ such that $|t(\theta)| \leqslant 1$ for all real θ. By an inequality of van der Corput and Schaake [2]

$$\{t'(\theta)\}^2 + (n-2)^2\,\{t(\theta)\}^2 \leqslant (n-2)^2 \qquad \text{for} \quad \theta \in \mathbb{R}.$$

Hence for $\theta \in \mathbb{R}$, we have

$$\begin{aligned}\{P'(\cos\theta)\}^2 &= \{t'(\theta)\sin\theta + 2t(\theta)\cos\theta\}^2 \\ &\leqslant \{t'(\theta)\}^2 + 4\{t(\theta)\}^2 \\ &\leqslant (n-2)^2 - (n^2-4n)\{t(\theta)\}^2\end{aligned}$$

which is equivalent to (1).

From (1) it follows. in particular, that $\|P'\| \leqslant n-2$. *Here* the restriction that "$P(x)$ is real for real x" can be dropped using standard reasoning. We may therefore state the following

COROLLARY 1. *If $P \in F_n^*$, then for $n \geqslant 4$*

$$\|P'\| \leqslant n-2. \tag{2}$$

Remark 1. If $P(x) := (1-x^2)\, T_{n-2}(x)$ then $P \in F_n^*$ and for *odd* $n \geqslant 5$

$$|P'(0)| = |T'_{n-2}(0)| = n-2.$$

Thus (2) is sharp at least for odd $n \geqslant 5$. It is also best possible for $n=4$ as the example $P(x) := (1-x^2)(2x^2-1)$ shows.

2.3. The estimate (2) can be improved for even $n \geqslant 6$. This follows from the next proposition and the fact that if $P \in F_n^*$, then [5, Theorem 1′]

$$|P'(0)| \leqslant n-3 \qquad \text{provided } n \text{ is even.} \tag{3}$$

PROPOSITION 2. *If $P \in F_n^*$, then*

$$|P'(x)| \leqslant \{(n-2)^2 - (n^2-4n)x^2\}^{1/2} \qquad \textit{for} \quad -1 \leqslant x \leqslant 1. \tag{4}$$

Proof. Let $\omega(z) := e^{i(n-2)z} \sin^2 z$. Then ω is an entire function of order 1 type n with only real zeros. Since its indicator function h_ω satisfies

$$h_\omega(-\pi/2) = n > -(n-4) = h_\omega(\pi/2)$$

it belongs to the class P introduced in [1, p. 129, see 7.8.2]. If we set $f(z) := P(\cos z)$ then the hypothesis implies that $|f(x)| \leqslant |\omega(x)|$ for $x \in \mathbb{R}$. Because f is an entire function of exponential type n we may apply Theorem 11.7.2 of [1] to conclude that $|f'(x)| \leqslant |\omega'(x)|$ for $x \in \mathbb{R}$. Hence for all real x, we have

$$\begin{aligned}|P'(\cos x)| &\leqslant |i(n-2)\sin x + 2\cos x| \\ &= \{(n-2)^2 - (n^2-4n)\cos^2 x\}^{1/2},\end{aligned}$$

and so (4) holds.

Remark 2. Inequality (4) shows, in particular, that for $n > 4$ the bound in (2) cannot be attained at a point $x \neq 0$.

2.4. In view of (3) and Proposition 2 it is natural to ask how large

$$\gamma_n := \sup_{P \in F_n^*} \|P'\| \tag{5}$$

can be if n is an even integer $\geqslant 6$. We prove

THEOREM 1. *For even n*

$$\gamma_n = n - 2 - \frac{\pi^2}{8n} + O(n^{-2}) \qquad as \quad n \to \infty. \tag{6}$$

A standard reasoning allows us to restrict ourselves to polynomials with real coefficients.

Throughout this sub-section, n will be supposed to be an even integer $\geqslant 6$.

The polynomial $P(x) := (1 - x^2)\, T_{n-2}(x)$ belongs to F_n^*. By a direct calculation we find

$$\left| P'\left(\frac{\pi}{2(n-2)}\right)\right| \geqslant n - 2 - \frac{\pi^2}{8n} + O(n^{-2}) \qquad \text{as} \quad n \to \infty.$$

Hence as a first step towards the proof of Theorem 1 we obtain

$$\gamma_n \geqslant n - 2 - \frac{\pi^2}{8n} + O(n^{-2}) \qquad \text{as} \quad n \to \infty. \tag{7}$$

Now for each $t \in [-1, 1]$ let us set

$$A_m(t) := \sup_{P \in F_m} |P'(t)|.$$

As the next step we prove:

LEMMA 1. *Let c be a fixed positive number and denote by I_c the interval $(0, \pi/2n - c/n^2)$. Then*

$$\gamma_n \leqslant \sup_{t \in I_c} (1 - t^2)\, A_{n-2}(t) + O(n^{-2}) \qquad as \quad n \to \infty. \tag{8}$$

Proof. Proposition 2 implies that if $P \in F_n^*$ then for $\pi/2n - c/n^2 \leqslant |x| \leqslant 1$

$$|P'(x)| \leqslant \left\{(n-2)^2 - (n^2 - 4n)\left(\frac{\pi}{2n} - \frac{c}{n^2}\right)^2\right\}^{1/2}$$

$$\leqslant n - 2 - \frac{\pi^2}{8n} + O(n^{-2}) \qquad \text{as} \quad n \to \infty.$$

Hence from (7) and the obvious symmetry we obtain

$$\gamma_n \leqslant \sup_{P \in F_n^*} \max_{0 \leqslant x \leqslant \pi/2n - c/n^2} |P'(x)| + O(n^{-2}) \qquad \text{as} \quad n \to \infty.$$

For each n let us choose $p_n \in F_n^*$ and x_n in $[0, \pi/2n - c/n^2]$ such that

$$\gamma_n \leqslant |p_n'(x_n)| + O(n^{-2}) \qquad \text{as} \quad n \to \infty. \tag{9}$$

Then, in view of (7) and Proposition 1, we must have

$$|p_n(x_n)| = O(n^{-1}) \qquad \text{as} \quad n \to \infty. \tag{10}$$

Writing $p_n(x) = (1 - x^2)\, q_{n-2}(x)$ we obtain

$$p_n'(x_n) = (1 - x_x^2)\, q_{n-2}'(x_n) - \frac{2x_n}{1 - x_n^2} p_n(x_n)$$

which, in conjunction with (10), implies that for $n \to \infty$

$$|p_n'(x_n)| = (1 - x_n^2)\, q_{n-2}'(x_n) + O(n^{-2}).$$

Since $q_{n-2} \in F_{n-2}$, we obtain

$$|p_n'(x_n)| \leqslant (1 - x_n^2)\, A_{n-2}(x_n) + O(n^{-2}).$$

Using this estimate in (9) we get the desired result.

Now we need to examine the function A_m quite closely. Its behaviour has been extensively studied (see [4, 8, 3, 5]) and much information is already available. However, to the best of our knowledge, the "convexity property" of A_m, contained in Lemma 2, which we need for our argument has not appeared in print before. Here are some of the known facts.

There is a unique polynomial $p(\cdot, t)$ (called extremal) in $\mathscr{P}_m$ with $\max_{-1 \leqslant x \leqslant 1} |p(x, t)| = 1$ such that

$$\left. \frac{\partial}{\partial x} p(x, t) \right|_{x=t} = A_m(t).$$

For certain values of t the extremal polynomials have been clearly identified. The zeros of the polynomials $(x+1)\, T_m''(x) + T_m'(x)$ and $(x-1)\, T_m''(x) + T_m'(x)$ are simple and lie in the interval $(-1, 1)$. If we denote them by $\xi_1 < \xi_2 < \cdots < \xi_{m-1}$ and $\eta_1 < \eta_2 < \cdots < \eta_{m-1}$, respectively, then

$$-1 < \xi_1 < \eta_1 < \xi_2 < \cdots < \eta_{m-2} < \xi_{m-1} < \eta_{m-1} < 1.$$

It is known that for t belonging to any of the intervals

$$[-1, \xi_1], [\eta_1, \xi_2], ..., [\eta_{m-2}, \xi_{m-1}], [\eta_{m-1}, 1]$$

(called Chebyshev intervals) the polynomial $p(\cdot, t)$ is either T_m or $-T_m$. In each of the complementary intervals (ξ_l, η_l), $l = 1, 2, ..., m-1$, there is a point ρ_l where T_{m-1} or $-T_{m-1}$ is extremal. The points

$$\lambda_l := \left(\sec^2 \frac{\pi}{2m}\right) \xi_l + \tan^2 \frac{\pi}{2m}$$

lie in (ξ_l, ρ_l) for $l = 1, 2, ..., m-1$ and at a point t belonging to the interval $(\xi_l, \lambda_l]$, $l = 1, 2, ..., m-1$, the extremal polynomial is

$$T_m\left(\frac{(1+\xi_l)(x-t)}{1+t} + \xi_l\right) \qquad \text{or} \qquad -T_m\left(\frac{(1+\xi_l)(x-t)}{1+t} + \xi_l\right).$$

Further, the points $\mu_l := (\sec^2(\pi/2m))\eta_l - \tan^2(\pi/2m)$ lie in (ρ_l, η_l) for $l = 1, 2, ..., m-1$ and the extremal polynomial at a point t belonging to $[\mu_l, \eta_l)$ is either $T_m((1-\eta_l)(x-t)/(1-t) + \eta_l)$ or $-T_m((1-\eta_l)(x-t)/(1-t) + \eta_l)$. Extremal polynomials corresponding to points belonging to intervals of the form (λ_l, ρ_l) or to those of the form (ρ_l, μ_l) are known to be Zolotarev polynomials. The intervals themselves are called (proper) Zolotarev intervals. Extremal polynomials corresponding to distinct values of t in the same Zolotarev interval are distinct. They are not easy to work with; however, it turns out that if m is even then $\rho_{m/2-1} = 0$ and $\mu_{m/2-1} = \pi/2m - (\pi^2/4 + 1)(1/m^2) + O(m^{-3})$ as $m \to \infty$. Now taking $m = n-2$ we deduce that for any $c > \pi^2/4 + 1 - \pi$ and all sufficiently large (even) integer n the interval I_c of Lemma 1 is contained in the Zolotarev interval $(\rho_{(n-2)/2-1}, \mu_{(n-2)/2-1}) = (0, \mu_{(n-2)/2-1})$. This is the reason why it is a bit hard to determine the supremum of $(1-t^2) A_{n-2}(t)$ for $t \in I_c$. In fact, we need the following.

LEMMA 2. *Let m be even. Then the restriction of A $(= A_m)$ to the interval $[0, \mu_{m/2-1})$ is an increasing two times continuously differentiable convex function.*

Proof. It follows from the investigations of Voronovskaja (see [8, Theorem 68; Remark, p. 166]) that $A'(0) = 0$ and $A'(t) > 0$ for $0 < t < \mu_{m/2-1}$. Hence $A(t)$ increases monotonically on $[0, \mu_{m/2-1})$ and attains its minimum value $m-1$ on $[0, \mu_{m/2-1})$ at $t = 0$. Besides, it has been shown by Gusev (see [8, pp. 193–195]) that A is two times continuously differentiable not only at the points of the interval $[0, \mu_{m/2-1})$

but throughout $[-1, 1]$ *except* at the points $(\xi_k)_{k=1}^{m-1}$, $(\eta_k)_{k=1}^{m-1}$, $(\lambda_k)_{k=1}^{m-1}$, and $(\mu_k)_{k=1}^{m-1}$. All we need to show is that

$$A''(t) \geqslant 0 \qquad \text{for} \quad 0 < t < \mu_{m/2-1}. \tag{11}$$

For this we shall use the ideas of W. A. Markov in the way they were presented in [5]. We recall that in [5] partial derivatives of a function $f(x, t)$ are denoted by

$$f_{j,k}(x, t) := \frac{\partial^{j+k}}{\partial x^j \, \partial t^k} f(x, t).$$

The more general function A given there reduces to the one considered here on setting $n = m$, $j = 1$, and $\lambda = \mu = 0$. In the notation of [5] we have (see the first and the third expressions for $A''(t)$ [5, p. 728])

$$A''(t) = p_{3,0}(t, t) - \frac{N}{2} d_0(t) \frac{F_{2,0}(t, t)}{\varphi_{1,0}(t, t)} F_{2,0}(t, t) \tag{12}$$

and

$$A''(t) = \frac{F_{2,0}(t, t)}{\varphi_{1,0}(t, t)} \frac{1}{\beta(t) - t} \times \left\{ A(t) \left(\frac{\Psi_{0,0}(t, t)}{F_{2,0}(t, t)} \frac{p_{3,0}(t, t)}{p_{1,0}(t, t)} + 2 \right) + 4tA'(t) \right\}. \tag{13}$$

We already know that

$$p_{1,0}(t, t) = A(t) > 0$$

and

$$A'(t) > 0 \qquad \text{for} \quad 0 < t < \mu_{m/2-1}. \tag{14}$$

We also need the following facts, namely (15)–(18). Since $\beta(t) \geqslant 1$ for even λ (see [5, pp. 716–717 or p. 730]) we have

$$\beta(t) - t > 0 \qquad \text{for} \quad 0 < t < \mu_{m/2-1}. \tag{15}$$

Further [5. p. 730]

$$\frac{\Psi_{0,0}(t, t)}{F_{2,0}(t, t)} \leqslant 0 \qquad \text{for} \quad 0 < t < \mu_{m/2-1} \tag{16}$$

and (see [5, p. 726, Formula (57)])

$$\frac{F_{2,0}(t, t)}{\varphi_{1,0}(t, t)} \geqslant 0 \qquad \text{for} \quad 0 < t < \mu_{m/2-1}. \tag{17}$$

Finally, by applying Lemma 6′ of [5] to the functions

$$g(x) = F(x, t) \qquad \text{and} \qquad h(x) = \frac{p(x, t)}{d_0(t)}$$

we obtain, as in [5, p. 729] (note the *misprint* in the third line from below; the inequality holds in the opposite direction),

$$F_{2,0}(t, t)\,\frac{p_{1,0}(t, t)}{d_0(t)} \leqslant 0 \qquad \text{for} \quad t \in (0, \mu_{m/2-1}). \tag{18}$$

Now we argue as follows. If $p_{3,0}(t, t) \geqslant 0$, then applying (17) and (18) we obtain the desired result from (12); but in the case $p_{3,0}(t, t) < 0$ the same conclusion follows from (13) in conjunction with (14), (15), (16), and (17).

2.5. *Completion of the proof of Theorem* 1. In Lemma 1 take $c = \pi^2/4$ $(> \pi^2/4 + 1 - \pi)$ and set $\alpha_n := \pi/2n - \pi^2/4n^2$. Then on I_c

$$A_{n-2}(t) \leqslant A_{n-2}(0) + \frac{A_{n-2}(\alpha_n) - A_{n-2}(0)}{\alpha_n}\, t$$

and

$$\sup_{t \in I_c} (1 - t^2)\, A_{n-2}(t)$$
$$\leqslant \sup_{t \in I_c} \left\{ A_{n-2}(0) + \frac{A_{n-2}(\alpha_n) - A_{n-2}(0)}{\alpha_n}\, t - A_{n-2}(0)\, t^2 \right\}.$$

Since

$$\frac{A_{n-2}(\alpha_n) - A_{n-2}(0)}{2\alpha_n A_{n-2}(0)} \sim \frac{1}{\pi} \qquad \text{as} \quad n \to \infty$$

we conclude that

$$\sup_{t \in I_c} (1 - t^2)\, A_{n-2}(t)$$
$$\leqslant A_{n-2}(\alpha_n) - A_{n-2}(0)\,\frac{\pi^2}{4n^2} + O(n^{-2})$$
$$\leqslant \frac{n-2}{\sqrt{1 - \alpha_n^2}} - A_{n-2}(0)\,\frac{\pi^2}{4n^2} + O(n^{-2}) \qquad \text{by Bernstein's inequality}$$
$$= n - 2 - \frac{\pi^2}{8n} + O(n^{-2}),$$

i.e.,

$$\gamma_n \leqslant n - 2 - \frac{\pi^2}{8n} + O(n^{-2}).$$

This, in conjunction with (7), implies (6) and the proof of Theorem 1 is complete.

3. The Derivative of a Polynomial Satisfying $0 \leqslant P(x) \leqslant (1-x^2)^{1/2}$ on $(-1, 1)$

If $P \in \mathscr{P}_n$ and $0 \leqslant P(x) \leqslant 1$ for $-1 \leqslant x \leqslant 1$ then the polynomial

$$f \colon x \mapsto 2P(x) - 1$$

belongs to F_n. The classical inequality of Markov may be applied to obtain

$$|P'(x)| = \tfrac{1}{2}|F'(x)| \leqslant \tfrac{1}{2}n^2 \qquad \text{for} \quad -1 \leqslant x \leqslant 1,$$

which is of course, well known. Thus requiring $P(x)$ to be non-negative on $[-1, 1]$ improves the bound for $\max_{-1 \leqslant x \leqslant 1} |P'(x)|$ by the factor $\frac{1}{2}$. If a polynomial $P \in \mathscr{P}_n$ satisfies $|P(x)| \leqslant (1-x^2)^{1/2}$ for $-1 \leqslant x \leqslant 1$, then [6]

$$|P'(x)| \leqslant 2(n-1) \qquad \text{for} \quad -1 \leqslant x \leqslant 1.$$

Shall we again get an improvement by the factor $\frac{1}{2}$ if we require $P(x)$ to be *non-negative* on $[-1, 1]$? Since we are assuming the graph of P on $[-1, 1]$ to lie inside the upper half D^+ of the unit disk it is reasonable to expect that an extremal polynomial "will oscillate between 0 and $(1-x^2)^{1/2}$" as often as the restriction on its degree will allow. The example which follows is "relevant" from this point of view.

If we denote by P_m the Legendre polynomial of degree m with the normalization $P_m(1) = 1$, then [7, p. 165, see (7.3.8)]

$$(1-x^2)^{1/4}\,|P_m(x)| < (2/\pi)^{1/2}\, m^{-1/2} \qquad \text{for} \quad -1 \leqslant x \leqslant 1.$$

Hence if n is even, then

$$P_*(x) := \frac{\pi}{2}\,\frac{n-2}{2}\,(1-x^2)\,P^2_{(n-2)/2}(x)$$

is a polynomial of degree n whose graph lies in D^+. Further, we note that

$$P'_*(1) = \frac{\pi}{2}(n-2).$$

This shows that the supremum M_n of $\|P'\|$ taken over all polynomials $P \in \mathscr{P}_n$ satisfying $0 \leqslant P(x) \leqslant (1-x^2)^{1/2}$ can be at least as large as $(\pi/2)(n-2)$; i.e., $M_n \geqslant (\pi/2)(n-2)$. We believe that

$$M_n = \frac{\pi}{2} n + \gamma_n \qquad \text{where} \quad n^{-1}\gamma_n \to 0 \quad \text{as} \quad n \to \infty \tag{19}$$

but we are able to prove much less. Our upper bound for M_n is contained in:

THEOREM 2. *If $P \in \mathscr{P}_n$ and $0 \leqslant P(x) \leqslant (1-x^2)^{1/2}$ for $-1 \leqslant x \leqslant 1$, then*

$$\|P'\| \leqslant \frac{1}{n-1} \sum_{\substack{k=1 \\ k \text{ odd}}}^{n-1} \left(1/\sin^2 \frac{(2k-1)\pi}{4(n-1)}\right). \tag{20}$$

For the proof of Theorem 2 we need the following

LEMMA 3. *Let*

$$l(x) := (x^2-1)\, T_{n-1}(x) = 2^{n-2}(x^2-1) \prod_{k=1}^{n-1} (x - x_k),$$

where $x_k := \cos((2k-1)\pi/2(n-1))$, $k = 1, ..., n-1$. Further, let $x_0 = 1$, $x_n = -1$ and for $k = 0, 1, ..., n-1$, n denote the quotient $l(x)/(x - x_k)$ by $l_k(x)$. Then $l'_k(x) \geqslant 0$ for $x \in [\cos(\pi/3(n-1)), 1]$ and $k = 0, 1, ..., n-1, n$.

Proof. Let $y_{n,1}$ denote the largest zero of l'_n. Then clearly $l'_n(x) \geqslant 0$ for all $x \geqslant y_{n,1}$. Further, if $y_{n,1} \leqslant x \leqslant 1$ then $l_n(x) < 0$ since all the zeros of l_n except 1 lie to the left of $y_{n,1}$. Since $l_k(x) = (x+1)\, l_n(x)/(x - x_k)$ we conclude that for $y_{n,1} \leqslant x \leqslant 1$,

$$l'_k(x) = \frac{(x+1)\, l'_n(x)}{x - x_k} - \frac{1 + x_k}{(x - x_k)^2}\, l_n(x) \geqslant 0$$

for $k = 0, 1, ..., n-1$ as well. It is now enough to show that $\cos(\pi/3(n-1)) \geqslant y_{n,1}$. For this we only need to check that $l'_n(\cos(\pi/3(n-1))) \geqslant 0$. But clearly $l'_n(\cos(\pi/3(n-1))) \geqslant 0$ if and only if

$$-\sqrt{3}\,(n-1) \sin \frac{\pi}{6(n-1)} + \cos \frac{\pi}{6(n-1)} \geqslant 0,$$

i.e., $\tan(\pi/6(n-1)) \leqslant 1/\sqrt{3}\,(n-1)$ which is true (since $\tan x \leqslant (2\sqrt{3}/\pi)x$ for $0 \leqslant x \leqslant \pi/6$).

Proof of Theorem 2. Let $(x_k)_{k=0}^{n}$ be as in Lemma 3. By the interpolation formula of Lagrange $P(x)=\sum_{k=0}^{n}(P(x_k)/l'(x_k))\,l_k(x)$ and so $P'(x) = \sum_{k=1}^{n-1}(P(x_k)/l'(x_k))l'_k(x)$. Since $l'(x_k)=(-1)^k(n-1)\sin((2k-1)\pi/2(n-1))$ we indeed have

$$P'(x)=\frac{1}{n-1}\sum_{k=1}^{n-1}\left((-1)^k P(x_k)\Big/\sin\frac{(2k-1)\pi}{2(n-1)}\right)l'_k(x).$$

Now let $\cos(\pi/3(n-1))\leqslant x\leqslant 1$. Using Lemma 3 and the fact that $0\leqslant P(x_k)\leqslant \sin((2k-1)\pi/2(n-1))$ we easily conclude that

$$|P'(x)|\leqslant\frac{1}{n-1}\sum_{\substack{k=1\\ k\text{ odd}}}^{n-1} l'_k(x).$$

Note that $l'_k(x)$ increases with x on the interval in question, i.e., $l'_k(x)\leqslant l'_k(1)$ and so

$$|P'(x)|\leqslant\frac{1}{n-1}\sum_{\substack{k=1\\ k\text{ odd}}}^{n-1}\left(1\Big/\sin^2\frac{(2k-1)\pi}{4(n-1)}\right). \tag{21}$$

Due to obvious symmetry the preceding estimate also holds for $-1\leqslant x\leqslant -\cos(\pi/3(n-1))$. In order to prove (21) for $|x|<\cos(\pi/3(n-1))$ we use the fact [6] that

$$|P'(x)|\leqslant\{(n-1)^2+x^2/(1-x^2)\}^{1/2}\qquad\text{for}\quad -1<x<1,$$

if $P\in\mathscr{P}_n$ and $|P(x)|\leqslant(1-x^2)^{1/2}$ for $-1<x<1$. This result shows that for $|x|<\cos(\pi/3(n-1))$ we have

$$|P'(x)|<\left\{(n-1)^2+\cot^2\frac{\pi}{3(n-1)}\right\}^{1/2}\leqslant\left(1+\frac{9}{\pi^2}\right)^{1/2}(n-1)$$
$$<\frac{1}{n-1}\sum_{\substack{k=1\\ k\text{ odd}}}^{n-1}\left(1\Big/\sin^2\frac{(2k-1)\pi}{4(n-1)}\right),$$

i.e., (21) holds for $|x|<\cos(\pi/3(n-1))$ as well. With this, the proof of Theorem 2 is complete.

Remark 3. It can be shown that if N_n denotes the right hand side of (21) and $G:=0\cdot 915965594177219015\cdots$ is Catalan's constant, then $N_n=(1+(8/\pi^2)G)n+O(1)$ as $n\to\infty$. Hence if $P\in\mathscr{P}_n$ and $0\leqslant P(x)\leqslant(1-x^2)^{1/2}$, then

$$\|P'\|<(1\cdot 7424537\cdots)n+O(1)\qquad\text{as}\quad n\to\infty \tag{22}$$

which, we admit, is a far cry from "$\|P'\|\leqslant(\pi/2)+o(1))n$ as $n\to\infty$".

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